Lecture 5. Modeling with First Order Equations

In this section, we will discuss about the applications of first-order equations.

Part 1. Polulation Models

Exponential Growth Model

Example 1.

A culture of yeast grows at a rate proportional to its size. If the initial population is 2000 cells and it doubles after 2 hours, answer the following questions.

(1) Write an expression for the number of yeast cells after t hours.

(2) Find the number of yeast cells after 7 hours.

(3) Find the rate at which the population of yeast cells is increasing at 7 hours.

ANS: (1) Let $P(t)$ be the population of yeast at time t.

Then we know.

sep

$$\frac{dP}{dt} = kP, \quad \underline{P(0)} = 2000, \quad P(2) = 2 \cdot P_0 = 4000$$

P_0

We have

$$\int \frac{dP}{P} = \int k\,dt \qquad e^{x+y} = e^x \cdot e^y$$

$$\Rightarrow \ln P = kt + C_1$$

$$\Rightarrow e^{\ln P} = e^{kt + C_1} = \cancel{e^{C_1}} \cdot e^{kt} = Ce^{kt}$$

C

So $P(t) = Ce^{kt}$ for some constants k. and C.

As $P(0) = 2000$, $P(0) = C \cdot e^{k \cdot 0} = C = 2000$

As $P(2) = 4000$. $P(2) = 2000\, e^{k \cdot 2} = 4000$

$\Rightarrow e^{k\cdot 2} = 2 \Rightarrow \ln e^{k\cdot 2} = \ln 2 \Rightarrow 2k = \ln 2$

$\Rightarrow k = \frac{\ln 2}{2}$.

Thus $P(t) = \underbrace{2000}_{P_0} e^{\overbrace{\frac{\ln 2}{2}}^{k} t}$

(2) $P(7) = 2000 e^{\frac{\ln 2}{2}\cdot 7} = 2000 e^{\frac{7}{2}\ln 2}$

(3) The question asks us $P'(7)$.

We know

$$P'(t) = \frac{dP}{dt} = kP(t) = \frac{\ln 2}{2} P(t) = \frac{\ln 2}{2} 2000 e^{\frac{\ln 2}{2} t}$$

Thus $P'(7) = 1000 \ln 2 \, e^{\frac{\ln 2}{2}\cdot 7}$

Summary: Exponential Growth Model

- Earlier we used the exponential differential equation

$$\frac{dP}{dt} = kP$$

with solution

$$P(t) = P_0 e^{kt}$$

to model natural population growth.

- This assumed that the birth and death rates were constant.
- Now we consider a more general population model that allows for nonconstant birth and death rates.

Variable Birth and Death Rates

- We define the birth rate function $\beta(t)$ as the number of births per unit of population per unit of time at time t.
- Similary, the death rate function $\delta(t)$ is the number of deaths per unit of population per unit of time at time t.
- Over the time interval $[t, t + \Delta t]$ there are then roughly $\beta(t) \cdot P(t) \cdot \Delta t$ births and $\delta(t) \cdot P(t) \cdot \Delta t$ deaths
- Thus the change in population over this time interval is

$$\Delta P = \{\text{ births }\} - \{\text{ deaths }\} \approx \beta(t) \cdot P(t) \cdot \Delta t - \delta(t) \cdot P(t) \cdot \Delta t$$

- Dividing by Δt gives

$$\frac{\Delta P}{\Delta t} \approx [\beta(t) - \delta(t)] P(t)$$

- Taking the limit as $\Delta t \to 0$ gives the **general population equation**

$$\frac{dP}{dt} = (\beta(t) - \delta(t)) P$$

- In the event that β and δ are constant, this equation reduces to the natural growth equation with $k = \beta - \delta$.
- But it also includes the possibility that β and δ vary with t.

The Logistic Equation

Decreasing Birth Rate

- We often observe that the birth rate of a population decreases as the population itself grows.
- One way to model this is to assume that the birth rate β is a linear decreasing function of the population size P, then

$$\beta = \beta_0 - \beta_1 P$$

  where β_0 and β_1 are positive constants.

- If the death rate $\delta = \delta_0$ remains constant, then our general population equation becomes

$$\frac{dP}{dt} = (\beta - \delta)P = (\beta_0 - \beta_1 P - \delta)P$$

- We can rewrite this as

$$\boxed{\frac{dP}{dt} = aP - bP^2}$$

  where $a = \beta_0 - \delta_0$ and $b = \beta_1$.

- If the coefficients a and b are both positive, then this equation is called the logistic equation.
- It is useful to rewrite the logistic equation in the form

$$\boxed{\frac{dP}{dt} = kP(M - P)}$$

  where $k = b$ and $M = a/b$ are constants.

Limiting Populations and Carrying Capacity

- The exponential differential equation has a general solution $P(t) = P_0 e^{kt}$. See Example 1 for a special case.
- It follows that

$$\lim_{t \to +\infty} P(t) = +\infty$$

- This means that the population grows without bound in a naturally growing population model.
- **Question:** If a population satisfies the logistic equation, what can we say about the population in the long-term?

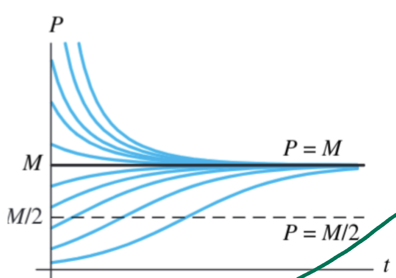
Example 2. Show that the solution of the logistic initial value problem

$$\frac{dP}{dt} = k\overset{P(t)}{P}(M - P), \quad P(0) = P_0$$

is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

Make it clear how your derivation depends on whether $0 < P_0 < M$ or $P_0 > M$.



ANS: If $P \neq 0$, and $M - P \neq 0$, then

$$\int \frac{dP}{P(M-P)} = \int k\,dt$$

Assume

$$\boxed{\frac{1}{P(M-P)}} = \frac{A}{P} + \frac{B}{M-P} = \frac{AM - AP + BP}{P(M-P)} = \boxed{\frac{(B-A)P + AM}{P(M-P)}}$$

Compaire the coefficients

$$\begin{cases} B - A = 0 \\ AM = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{M} \\ B = \frac{1}{M} \end{cases}$$

$$\frac{1}{M}\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt$$

$$\Rightarrow \frac{1}{M}\left(\int \frac{1}{P}\,dP - \int \frac{d(M-P)}{M-P}\right) = kt + C_1$$

$$\Rightarrow \frac{1}{M}\left(\ln|P| - \ln|M-P|\right) = kt + C_1$$

$$\Rightarrow \ln\left|\frac{P}{M-P}\right| = Mkt + \overset{C_2}{\cancel{MC_1}}$$

$$\Rightarrow \frac{P}{M-P} = \overset{C}{\cancel{\pm e^{C_2}}} \cdot e^{Mkt}$$

$$\Rightarrow \frac{P(t)}{M - P(t)} = C e^{Mkt}$$

Since $P(0) = P_0$, we know $\frac{P_0}{M - P_0} = C\cancel{e^{0}}^{1} = C$

we have $\frac{P(t)}{M - P(t)} = \frac{P_0}{M - P_0} e^{Mkt}$

$$\Rightarrow \quad P(M - P_0) = P_0(M - P)e^{Mkt}$$

$$\Rightarrow \quad \underline{P}(M - P_0) + \underline{P} P_0 e^{Mkt} = P_0 M e^{Mkt}$$

$$\Rightarrow \quad P\left[(M - P_0) + P_0 e^{Mkt}\right] = P_0 M e^{Mkt}$$

$$\Rightarrow \quad P(t) = \frac{P_0 M e^{Mkt} \cdot e^{-Mkt}}{\left[(M - P_0) + P_0 e^{Mkt}\right] \cdot e^{-Mkt}}$$

$$\Rightarrow \quad P(t) = \frac{M P_0}{P_0 + (M - P_0)e^{-Mkt}}$$

- If $P_0 = M$. we have $P(t) = \frac{M \cdot M}{M + 0} = M$.
- If $0 < P_0 < M$, $\frac{dP}{dt} = kP(M - P) > 0$ near P_0 and

$$P(t) = \frac{P_0 M}{P_0 + \{\text{pos. number}\}} < \frac{P_0 M}{P_0} = M$$

- If $P_0 > M$, $\frac{dP}{dt} = kP(M - P) < 0$ near P_0 and

$$P(t) = \frac{P_0 M}{P_0 + \{\text{negative. number}\}} > \frac{P_0 M}{P_0} = M$$

- In either case, we find $\lim_{t \to \infty} P(t) = \frac{M P_0}{P_0 + 0} = M$
- $P(t)$ approaches the finite limiting population M as $t \to \infty$
- We sometimes call M the carrying capacity of the environment.

Example 3. Assuming $P \geq 0$, suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.07P - 0.0007P^2$$

where t is measured in weeks. Answer the following questions.

1. The carrying capacity is the limit $\lim_{t\to\infty} P(t)$ of the population size after a very long time. What is the carrying capacity?

M in Eg 2.

$$\frac{dP}{dt} = 0.0007P(\underline{100} - P)$$

Thus the carrying capacity is 100.

2. When P is very small its growth is approximately exponential: $P(t) \approx Ae^{kt}$ for some constants A and k. Here k represents the "exponential growth rate". In this problem what is the value of k?

$$\frac{dP}{dt} = 0.07P - \cancel{0.0007P^2}$$

when P is small we can ignore the term $0.0007P^2$

$k = 0.07$

3. For what values of P is the population increasing?

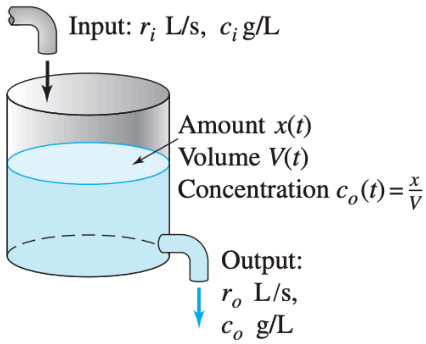
From the Eg 2. we know $P \in (0, 100)$

4. For what values of P is the population decreasing?

$P \in (100, \infty)$

Part 2. An Application of Linear First-Order Equations: Mixture Problems

- A tank containing a solution-a mixture of solute and solvent-has both inflow and outflow. Our goal is to find the amount $x(t)$ of solute at time t, given the initial amount x_0.
- Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the (mixed) solution in the tank flows out at the rate of r_o liters per second.



Analysis: Set up a differential equation for $x(t)$

- We want to estimate the change Δx in x during the brief time interval $[t, t + \Delta t]$.
- The amount of solute that flows into the tank during Δt seconds is $r_i c_i \Delta t$ grams.
- The amount that flows out of the tank is more complex because it depends upon the conceration $c_o(t) = \frac{x(t)}{V(t)}$ of solute in the solution at time t
- So the change Δx in the amount of solute is:

$$\Delta x = \{\text{grams input}\} - \{\text{grams output}\} \approx r_i c_i \Delta t - r_o c_o \Delta t$$

- Dividing by Δt, gives

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o$$

- Let $\Delta t \to 0$,

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

- Note r_i, c_i, and r_o are constant. But $c_o(t) = \frac{x(t)}{V(t)}$.
- If $V_0 = V(0)$, then $V(t) = V_0 + (r_i - r_o)t$ So $V(t)$ is a constant when $r_i = r_o$.

Therefore,

$$\boxed{\frac{dx}{dt} = r_i c_i - r_o \frac{x(t)}{V(t)}, \text{ where } V(t) = V_0 + (r_i - r_o)t}$$

Note this is a linear 1st order eqn.

$V(0)$ $\shortparallel$ $x(0) = 0$

Example 4

A tank initially contains 240 gal of pure water. Brine containing $1/4\mathrm{lb}$ of salt per gallon enters the tank at $2\mathrm{gal}/\mathrm{min}$, and the (perfectly mixed) solution leaves the tank at $4\mathrm{gal}/\mathrm{min}$; thus the tank is empty after exactly $2\ \mathrm{h}$.

(a) Find the amount of salt in the tank after t minutes. $x(t)$

(b) What is the maximum amount of salt ever in the tank?

ANS: $V(0) = V_0 = 240$ gal

$r_i = 2\text{gal}/\text{min}$, $\quad C_i = \frac{1}{4}\ \text{lb}/\text{gal}$.

$r_o = 4\text{gal}/\text{min}$. $\quad C_o = \dfrac{x(t)}{V(t)}$. where $V(t) = V_0 + (r_i - r_o)t$

$= 240 - 2t$

We have $\dfrac{dx}{dt} = r_i C_i - r_o C_o$ where $C_o = \dfrac{x(t)}{V(t)}$

$$\Rightarrow \frac{dx}{dt} = 2 \cdot \frac{1}{4} - 4 \cdot \frac{x(t)}{240-2t}$$

$$\Rightarrow \frac{dx}{dt} + \underbrace{\frac{2}{120-t}}_{P(t)} x(t) = \underbrace{\frac{1}{2}}_{Q(t)} \qquad ①$$

Exercise 5. A tank contains 70 kg of salt and 1000 L of water. Pure water enters a tank at the rate $8 \text{ L}/\text{min}$. The solution is mixed and drains from the tank at the rate $4 \text{ L}/\text{min}$.

V_0 $\quad C_i = 0 \text{ kg}/L$ $\quad r_i = 8 \text{ L}/\text{min}$

(1) What is the amount of salt in the tank initially? $\quad r_o = 4 \text{ L}/\text{min}$

(2) Find the amount of salt in the tank after 2 hours. $= 120 \text{ min}$

(3) Find the concentration of salt in the solution in the tank as time approaches infinity.

ANS: (1) We know $x(0) = 70 \text{ kg}$ from the question

(2) We need to find $x(120)$. So we have to set up an eqn for $x(t)$ and solve it.

We know $x(0) = 70 \text{ kg}$. $C_i = 0 \text{ kg}/L$,

$r_i = 8 \text{ L}/\text{min}$, $\quad r_o = 4 \text{ L}/\text{min}$

$C_o = \dfrac{x(t)}{V(t)}$, where $V(t) = V_0 + (\overset{8}{\cancel{r_i}} - \overset{4}{\cancel{r_o}})t = 1000 + 4t$

Thus

$$\frac{dx}{dt} = r_i C_i - r_o \frac{x(t)}{V(t)}$$

$$= 8 \cdot 0 - 4 \frac{x}{1000 + 4t}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{x}{1000 + 4t}$$

(separable eqn in this special case).

$$\Rightarrow \int \frac{dx}{x} = \int -\frac{1}{250 + t} \, dt$$

$$\Rightarrow \ln x = -\ln(250 + t) + C_1 = \ln(250 + t)^{-1} + C_1$$

$$\Rightarrow x(t) = e^{\ln(250+t)^{-1}} \cdot e^{C_1} = C(250 + t)^{-1}$$

$$\Rightarrow \quad x(t) = \frac{C}{250+t}$$

As $x(0) = 70$, we have $\frac{C}{250+0} = 70$

$$\Rightarrow C = 250 \times 70 = 17500$$

Thus $x(t) = \frac{17500}{250+t}$

Note 2 hours = 120 mins.

We have $X(120) = \frac{17500}{250+120} = \frac{17500}{370} = \frac{1750}{37}$ kg

(3) We have $x(t) = \frac{17500}{250+t}$

Thus $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{17500}{250+t} = 0$

Thus the amount of salt in the tank goes to 0 as $t \to \infty$.

So the concentration $C(t) \left(= \frac{X(t)}{V(t)}\right)$ goes to 0 kg/L

Exercise 6. At time $t = 0$, a tank contains 20oz of salt dissolved in 100 gallons of water. Then brine containing 6oz of salt per gallon of brine is allowed to enter the tank at a rate of $5\text{gal}/\text{min}$ and the mixed solution is drained from the tank at the same rate.

(1) How much salt is in the tank at an arbitrary time?

(2) How much salt is in the tank at time 25 min?

ANS: We know $x(0) = 20$ oz, $V(0) = V_0 = 100$ gal

$r_i = 5\text{gal/min}$, $c_i = 6$ oz/gal

$r_o = 5\text{gal/min}$.

Then $V(t) = V_0 + (r_i - r_o)t = V_0 = 100$ gal.

Thus

$$\frac{dx}{dt} = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x(t)}{V(t)}$$

$$= 30 - 5 \cdot \frac{x}{100}$$

$$\Rightarrow \frac{dx}{dt} = 30 - \frac{x}{20}, \quad x(0) = 20. \ \circledast$$

(separable eqn) & linear first order.

(1) Feel free to solve the eqn either using method for separable or linear first order.

① Solving $\circledast$ as a separable eqn: Note $\frac{dx}{dt} = \frac{600 - x}{20}$

Warning: There is a negative sign here

$$\Rightarrow \int \frac{dx}{600 - x} = \int \frac{dt}{20}$$

$$\Rightarrow \int \frac{-d(600-x)}{600-x} = \int \frac{dt}{20}$$

$$\Rightarrow -\ln(600-x) = \frac{1}{20}t + C_1$$

$$\Rightarrow \ln(600-x) = -\frac{1}{20}t - C_1$$

$$\Rightarrow 600 - x = Ce^{-\frac{1}{20}t}$$

As $x(0)=20$, we have

$$600-20 = Ce^{-\frac{1}{20}\cdot 0} \Rightarrow C = 580$$

Thus $$600-x = 580\, e^{-\frac{1}{20}t}$$

$$\Rightarrow x(t) = 600 - 580\, e^{-\frac{1}{20}t}$$

⊖ Solving ⊛ as linear first order eqn:

We have $\frac{dx}{dt} + \frac{x}{20} = 30$, $x(0)=20$.

An integrating factor is $\rho = e^{\int \frac{1}{20}dt} = e^{\frac{t}{20}}$

Multiply both sides by the ρ, we have

$$\frac{d}{dt}(\rho x) = 30\cdot e^{\frac{t}{20}}$$

Integrating, we get

$$\rho x = e^{\frac{t}{20}}\cdot x(t) = \int 30 e^{\frac{t}{20}} dt = 30\times 20 \int e^{\frac{t}{20}} d\frac{t}{20}$$

$$\Rightarrow e^{\frac{t}{20}} x(t) = 600 e^{\frac{t}{20}} + C_2$$

$$\Rightarrow x(t) = 600 + C_2 e^{-\frac{t}{20}}$$

As $x(0)=20$, $x(0) = 600 + C_2\cdot e^0 = 20$

$\Rightarrow C_2 = -580$

Thus $x(t) = 600 - 580\, e^{-\frac{t}{20}}$

Note both methods give the same result.

(2). Amount of salt after 25 minutes is

$$x(25) = 600 - 580\, e^{-\frac{25}{20}} \approx 433.8 \text{ oz}.$$

Exercise 7. A tank initially contains 200 gallons of brine, with 50 pounds of salt in solution. Brine containing 2 pounds of salt per gallon is entering the tank at the rate of 4 gallons per minute and is is flowing out at the same rate. If the mixture in the tank is kept uniform by constant stirring, find the amount of salt in the tank at the end of 40 minutes.

Solution. From the question, we know $x(0) = 50$ pounds, $V(0) = 200$ gallons, $r_i = 4$ gallons/min, $c_i = 2$ pounds/gallon, and $r_o = 4$ gallons/min.

Thus $V = V_0 + (r_i - r_o)t = V_0 = 200$, which is a constant.

So we have

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 8 - 4\frac{x(t)}{100}$$

Simplify, we have

$$\frac{dx}{dt} = 8 - \frac{x(t)}{25}$$

You can either solve it as a separable equation or a linear first order equation. We will solve it as a separable equation in this note. We have

$$\frac{dx}{dt} = \frac{200 - x(t)}{25}$$

Thus

$$\int \frac{1}{x - 200} dx = -\frac{1}{25} \int dt$$

Thus we have $\ln|x - 200| = -\dfrac{1}{25}t + C$.

As $x(0) = 50$, we know $C = \ln 150$.

Thus $\ln|x - 200| = -\dfrac{1}{25}t + \ln 150 \implies |x - 200| = e^{\ln 150 - \frac{1}{25}t} = 150e^{-\frac{1}{25}t}$

Note when t is close to $t = 0$, $x(t)$ is close to 50 pounds, which is less than 200. So we may assume $x - 200 < 0$ when we remove the absolution sign.

Thus we have $|x - 200| = 200 - x = 150e^{-\frac{1}{25}t}$
So $x = 200 - 150e^{-\frac{1}{25}t}$.
Therefore $x(40) = 200 - 150e^{-\frac{40}{25}} \approx 169.716$ pounds.

Exercise 8. A population P obeys the logistic model. It satisfies the equation

$$\frac{dP}{dt} = \frac{9}{1100} P(11 - P) \text{ for } P > 0$$

1. The population is increasing when ___$< P <$___
2. The population is decreasing when $P >$ ___
3. Assume that $P(0) = 2$. Find $P(40)$.

Solution.

1. From the discussion in Example 2, we know the population is increasing when $0 < P < 11$.
2. The population is decreasing when $P > 11$.
3. You can use either the formula derived in Example 2 directly, or compute the solution yourself.

   Note if you choose to apply the formula directly, we have $P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}$, with $k = \dfrac{9}{1100}$, $M = 11$ and $P(0) = 2$.

   Thus

   $P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} = \dfrac{22}{2 + 9e^{-9t/100}}$.

   Then $P(40) = \dfrac{22}{2 + \dfrac{9}{e^{18/5}}} \approx 9.79557$

   To solve this equation as a separable differential equation, we rewrite it as

$$\int \frac{1}{P(11 - P)} dP = \int \frac{9}{1100} dt$$

Assume $\dfrac{1}{P(11 - P)} = \dfrac{A}{P} + \dfrac{B}{11 - P} = \dfrac{11A - AP + BP}{P(11 - P)} = \dfrac{11A + (B - A)P}{P(11 - P)}$.

Compare the coefficients, we know

$B - A = 0$ and $11A = 1$. Thus $A = B = \dfrac{1}{11}$

Thus

$$\frac{1}{11}\left(\int \frac{1}{P} dP + \int \frac{1}{11 - P} dP\right) = \int \frac{9}{1100} dt$$
$$\implies \left(\int \frac{1}{P} dP - \int \frac{d(11 - P)}{11 - P}\right) = 11 \int \frac{9}{1100} dt$$

Thus we have

$$\ln|P| - \ln|11-P| = \ln\left|\frac{P}{11-P}\right| = \frac{9}{100}t + C_1$$

So

$$\frac{P}{11-P} = Ce^{\frac{9}{100}t}$$

As $P(0) = 2$, we know $\frac{2}{9} = C$

Thus

$$\frac{P}{11-P} = \frac{2}{9}e^{\frac{9}{100}t}$$

Solve for $P(t)$, we get $P(t) = \dfrac{22e^{(9t)/100}}{2e^{(9t)/100} + 9}$

Then we get $P(t) = \dfrac{22e^{18/5}}{9 + 2e^{18/5}} \approx 9.79557$, which is the same as we got by applying the formula.